Note Title 1/16/2011

فصل اول : مدارهای منطقی ترکیبی

$$Z = \sum_{C} e^{-\beta H}$$

C = Configurations

$$Z = e^{-\beta F}$$

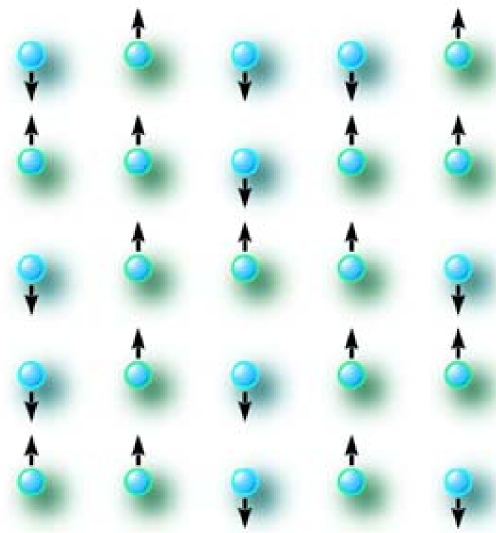
$$F = \frac{-1}{\beta} \ln Z = -kT \ln Z$$

$Z \longrightarrow$ All thermodynamics.

$$U = \langle H \rangle = \frac{1}{Z} \sum_{C} e^{-\beta H} H = \frac{-\partial}{\partial \beta} \ln Z.$$

$$F = U - TS \longrightarrow S = \frac{U - F}{T}$$

Ising Model



$$H = -J \sum_{\langle i,j \rangle} S_i S_j \qquad S_i = \pm 1$$

N = تعداد سایت‌ها

2^N = تعداد پیکربندی‌ها

$$Z = \sum_{\{s\}} e^{\beta J \sum_{\langle i,j \rangle} s_i s_j}$$

One-dimensional Ising Model

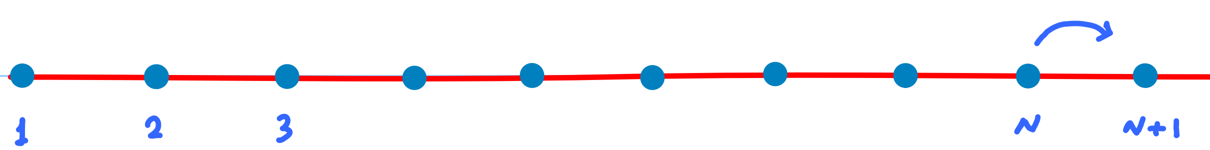
Method 1) Recursive Method

$$H = -\sum_{k=1}^{N-1} J_k \sigma_k \sigma_{k+1}$$

$$Z = \sum_{C} e^{\beta \sum_{k=1}^{N-1} J_k \sigma_k \sigma_{k+1}}$$

$$\beta J_k = \mathcal{J}_k$$

$$Z_N = \sum_{\sigma_1 \dots \sigma_N} e^{\sum_{k=1}^{N-1} \mathcal{J}_k \sigma_k \sigma_{k+1}}$$



$$Z_{N+1} = \sum_{\sigma_1 \dots \sigma_N, \sigma_{N+1}} e^{\sum_{k=1}^{N} \mathcal{J}_k \sigma_k \sigma_{k+1}}$$

$$Z_{N+1} = \sum_{\sigma_1 \cdots \sigma_N,\ \sigma_{N+1}} e^{\sum_{k=1}^{N-1} J_k \sigma_k \sigma_{k+1} + J_N \sigma_N \sigma_{N+1}}$$

$$Z_{N+1} = \sum_{\sigma_1 \cdots \sigma_N} e^{\sum_{k=1}^{N-1} J_k \sigma_k \sigma_{k+1}} \sum_{\sigma_{N+1}} e^{J_N \sigma_N \sigma_{N+1}}$$

$$\sum_{\sigma_{N+1}=1}^{-1} e^{J_N \sigma_N \sigma_{N+1}} = e^{J_N \sigma_N} + e^{-J_N \sigma_N}$$

$$= 2 \cosh J_N$$

مهمترین نکته در روش تکرار این است که این جمع به هیچ مشکلی ندارد.

$$\rightarrow \quad Z_{N+1} = Z_N \left(2\cosh J_N\right)$$

$$\rightarrow \quad Z_{N+1} = Z_{N-1} \; 2\cosh J_{N-1} \; 2\cosh J_N$$

$$\cdots\cdots$$

$$\boxed{Z_{N+1} = \prod_{k=1}^{N} \left(2\cosh J_k\right)}$$

برای یک سیستم همگن :

$$J_k = J$$

$$Z_N = \left(2\cosh \beta J\right)^{N-1}$$

$$\langle \sigma_k \sigma_{k+1} \rangle = \frac{1}{Z_N} \sum_{\sigma_1 \dots \sigma_N} \sigma_k \sigma_{k+1} \, e^{\sum_{i=1}^{N-1} J_i \sigma_i \sigma_{i+1}}$$

$$\rightarrow \qquad \langle \sigma_k \sigma_{k+1} \rangle = \frac{1}{Z_N} \frac{\partial}{\partial J_k} Z_N$$

$$\langle \sigma_k \sigma_{k+2} \rangle = \frac{1}{Z_N} \sum_{\{\sigma\}} \underbrace{\sigma_k \sigma_{k+1}} \, \underbrace{\sigma_{k+1} \sigma_{k+2}} \, e^{-\beta H}$$

$$= \frac{1}{Z_N} \frac{\partial}{\partial J_k} \frac{\partial}{\partial J_{k+1}} Z_N$$

$$\langle \sigma_k \sigma_{k+1} \cdots \sigma_{k+r} \rangle = \frac{1}{Z_N} \frac{\partial}{\partial J_k} \cdots \frac{\partial}{\partial J_{k+r}} Z_N$$

$$Z_N = \prod_{i=1}^{N-1} (2\cosh J_i)$$

$$\frac{\partial}{\partial J_1} \cosh J_1 = \sinh J_1$$

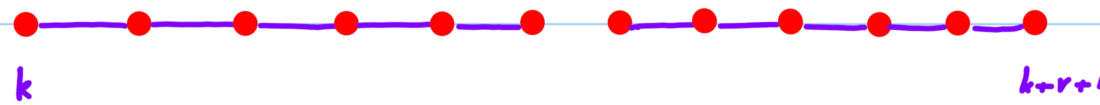
$$\frac{1}{\cosh J_1 \cosh J_2 \cosh J_3} \frac{\partial}{\partial J_1} \frac{\partial}{\partial J_2} \cosh J_1 \cosh J_2 \cosh J_3$$

$$= \frac{1}{\cosh J_1 \cosh J_2 \cosh J_3} \sinh J_1 \sinh J_2 \cosh J_3$$

$$\boxed{\langle \sigma_k \sigma_{k+1} \dots \sigma_{k+r} \rangle = \prod_{i=k}^{k+r} \tanh J_i}$$

اگر هر کدام از اتصالات پاره شود یعنی $J_i \to 0$ ⇐ $\langle \dots \rangle = 0$

$J=0$



$$\langle \sigma_k \cdots \sigma_{k+r+1} \rangle = 0.$$

اگر سیستم همگن باشد . $J_k = J$.

$$G^{(2)}(r) = \langle \sigma_k \sigma_{k+r} \rangle = (\tanh J)^r$$

$$= e^{r \ln(\tanh J)} = e^{-r \ln \frac{1}{\tanh J}}$$

$$\rightarrow \boxed{\langle \sigma_k \sigma_{k+r} \rangle = e^{-r/\xi}}$$

$$\xi = \frac{-1}{\ln(\tanh J)}$$

$$\xi = \frac{1}{\ln\left(\coth \frac{J}{kT}\right)}$$

$$\left\{\begin{array}{l} T \to 0 \\ J \to \infty \end{array}\right. \qquad \xi \to \infty$$

$$\left\{\begin{array}{l} T \to \infty \\ J \to 0 \end{array}\right. \qquad \xi \to 0$$

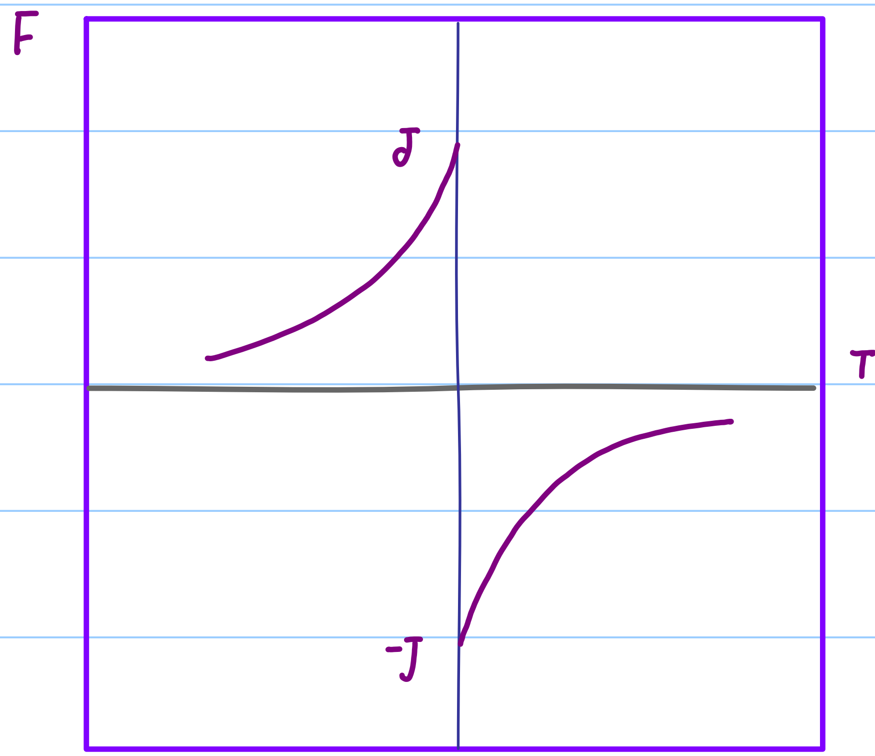
Free Energy $\quad F = -kT \ln Z$

$$\boxed{f = \frac{F}{N} = -kT \ln\left(2 \cosh \frac{J}{kT}\right)}$$

تابع انرژی آزاد f ، تنها در نقطه $T=0$ غیر تحلیلی است.

$$T \to 0^+ \qquad f \to -kT \ln\left(e^{J/kT}\right) = -kT\frac{J}{kT} = -J$$

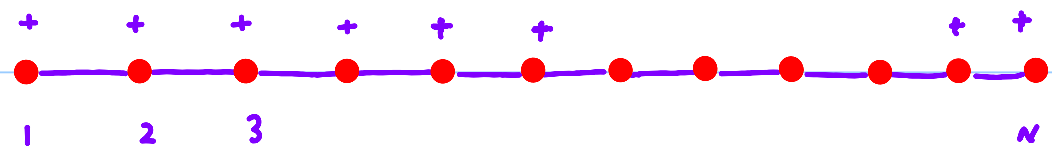
$$T \to 0^- \qquad f \to -kT \ln\left(e^{-J/kT}\right) = kT\frac{J}{kT} = J$$



پس در هیچ دمای محدودی گذار فاز، به معنای فیزیکی خود رخ نمی دهد.

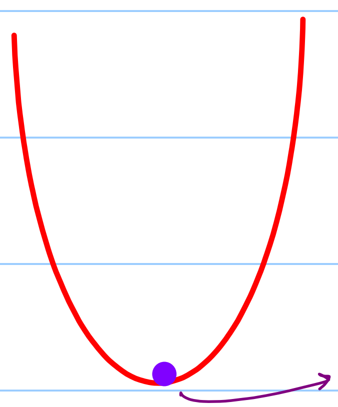
• استدلال فیزیکی برای اینکه چرا در دمای محدود گذار فاز رخ نمی دهد. (البته در مدل یک بعدی).

ordered phase ب

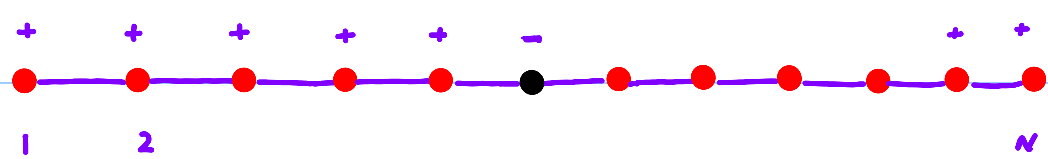


$$E_0 = -J(N-1)$$

سوال اینکه: آیا این فاز منظم پایدار است؟



the ordered phase??

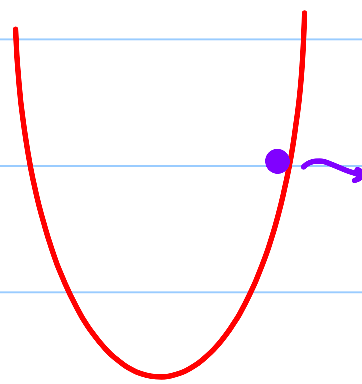


$$\Delta E = +4J$$

$$\Delta S = +k \ln N$$

$$\Delta F = \Delta E - T\Delta S$$

$$= 4J - kT \ln N$$



the ordered phase

- انرژی متوسط

$$U = -\frac{\partial}{\partial\beta} \ln Z = -\frac{\partial}{\partial\beta} \ln \prod_{i=1}^{N-1} (2\cosh\beta J_i)$$

$$= -\frac{\partial}{\partial\beta} \sum_{i=1}^{N-1} (\ln 2 + \ln\cosh\beta J_i)$$

$$= \sum_{i=1}^{N-1} -J_i \tanh\beta J_i$$

$$\boxed{\mathcal{U} = -(N-1) J \tanh \beta J}$$

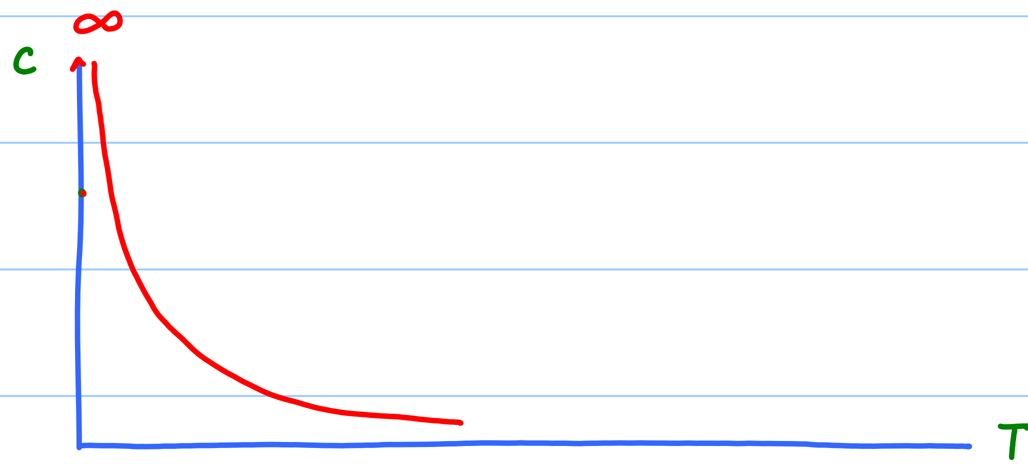
$$u = -J \tanh \frac{J}{kT}$$

• ظرفیت گرمایی

$$c(B=0,T) = \frac{\partial u}{\partial T} = -J \frac{\partial}{\partial T} \tanh \frac{J}{kT}$$

$$= -J \frac{\left(-J/kT^2\right)}{\cosh^2 \frac{J}{kT}}$$

$$\rightarrow \quad C = k \frac{x^2}{\cosh^2 x} \qquad x = \frac{J}{kT}$$



- پذیرفتاری مغناطیسی $\quad \chi$

Magnetic Susceptibility

$$\chi(T, B=0) = \frac{\partial}{\partial B} M$$

$$H = H_I(\sigma_1 \dots \sigma_N) - B \sum_i \sigma_i$$

$$Z = \sum_{\{\sigma\}} e^{-\beta H_I + \beta B \sum_i \sigma_i}$$

$$M = \frac{1}{Z}\frac{1}{\beta}\frac{\partial}{\partial B} Z$$

$$\frac{\partial M}{\partial B} = \frac{1}{\beta}\left(\frac{-1}{Z^2}\frac{\partial Z}{\partial B}\right)\frac{\partial Z}{\partial B} + \frac{1}{\beta}\frac{1}{Z}\frac{\partial^2 Z}{\partial B^2}$$

$$= -\beta\left(\sum_i \langle \sigma_i \rangle\right)^2 + \beta \sum_{i,j} \langle \sigma_i \sigma_j \rangle$$

$$= \beta \sum_{i,j} \langle \sigma_i \sigma_j \rangle - \beta \sum_{i,j} \langle \sigma_i \rangle \langle \sigma_j \rangle$$

$$\boxed{\chi = \frac{\partial M}{\partial B} = \beta \sum_{i,j} G_c(i,j)}$$

$$\chi = \beta \int dr\, dr'\, G_c(r-r')$$ برای سیستم های پیوسته

$$= \beta V \int dr\, G_c(r).$$

محاسبهٔ χ برای مدل یک بعدی.

$$\langle \sigma_i \sigma_j \rangle = v^{|i-j|} \qquad v = \tanh \beta J$$

$$\begin{vmatrix} 1 & v & v^2 & & & & & \\ v & 1 & v & v^2 & & & & \\ v^2 & v & 1 & v & v^2 & & & \\ & v^2 & v & 1 & v & & & \\ & & v^2 & v & 1 & & & \\ & & & & & \ddots & & v^2 \\ & & & & & v & \cdot & v \\ & & & & & & v & 1 \end{vmatrix}$$

$$\chi(T, B=0) = \frac{\beta}{N} \Big\{ N + 2(N-1)v + 2(N-2)v^2 + \cdots$$
$$2(N-(N-1))\,v^{N-1} \Big\}$$

$$= \frac{\beta}{N} \left\{ N + 2 \sum_{k=1}^{N-1} (N-k)\, v^k \right\}$$

$$\sum_{k=1}^{N-1} v^k = \frac{v + v^N}{1 - v}$$

$$\sum_{k=1}^{N-1} k v^k = v \sum_{k=1}^{N-1} \frac{\partial}{\partial v} \left(v^k \right) = v \frac{\partial}{\partial v} \; \frac{v + v^N}{1 - v}$$

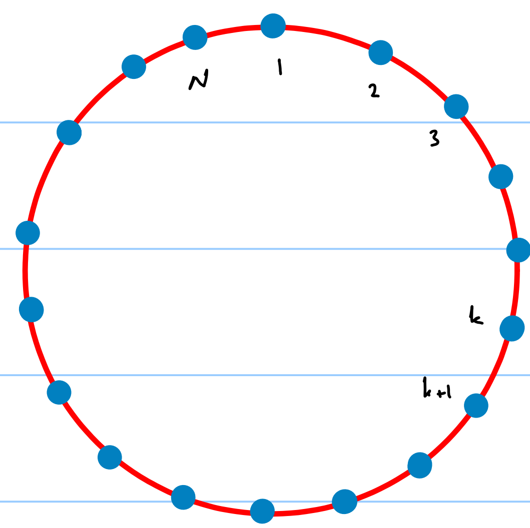
← جایگذاری

$$\chi(T, B=0) = \beta \, \frac{1+v}{1-v}$$

$$\chi(T, B=0) = \beta \, \frac{1 + \tanh \beta J}{1 - \tanh \beta J}$$

$$\boxed{\chi(T, B=0) = \beta \, e^{\beta J} = \frac{1}{kT} \, e^{\frac{J}{kT}}}$$

- **Method 2)** Transfer Matrix Method



$$H = -\sum_{i=1}^{N} J\,\sigma_i\,\sigma_{i+1} - B\sum_{i=1}^{N}\sigma_i$$

حسن این روش: حل مدل با حضور میدان مغناطیسی.

$$Z_N = \sum_{\sigma_1 \dots \sigma_N} \prod_{i=1}^{N} e^{\beta J \sigma_i \sigma_{i+1} + \beta B \sigma_i}$$

$$= \sum_{\sigma_1 \dots \sigma_N} \langle \sigma_1 | V | \sigma_2 \rangle \langle \sigma_2 | V | \sigma_1 \rangle \dots \langle \sigma_N | V | \sigma_1 \rangle$$

$$\langle \sigma | V | \sigma' \rangle = e^{\beta J \sigma \sigma' + \beta \frac{B}{2} (\sigma + \sigma')}$$

$$V = \begin{bmatrix} e^{\beta (J + B)} & e^{-\beta J} \\ e^{-\beta J} & e^{\beta (J - B)} \end{bmatrix}$$

$$\rightarrow \quad Z_N = tr(V^N) = \lambda_+^N + \lambda_-^N$$

Thermodynamic Limit $N \to \infty$

$$Z_N \to \lambda_+^N \left(1 + \left(\frac{\lambda_-}{\lambda_+}\right)^N\right) \to \lambda_+^N$$

$$\to \quad F = -kT \ln Z = -NkT \ln \lambda_+$$

$$f = \frac{F}{N} = -kT \ln \lambda_+$$

$$\to \quad V = \begin{bmatrix} e^{\beta(J+B)} & e^{-\beta J} \\ e^{-\beta J} & e^{\beta(J-B)} \end{bmatrix}$$

پیدا کردن ویژه مقدارهای ماتریس انتقال:

$$\left(e^{\beta(J+B)}-\lambda\right)\left(e^{\beta(J-B)}-\lambda\right)-e^{-2\beta J}=0$$

$$2\sinh 2\beta J+\lambda^{2}-2e^{\beta J}\cosh\beta B\,\lambda=0$$

$$\lambda_{\pm}=e^{\beta J}\cosh\beta B\pm\sqrt{e^{2\beta J}\cosh^{2}\beta B-2\sinh 2\beta J}$$

$$f(\beta)=-kT\ln\lambda_{+}$$

$$F(\beta,B)=-\frac{1}{\beta}\ln\left[e^{\mathcal{J}}\cosh\mathcal{B}+\sqrt{e^{2\mathcal{J}}\cosh^{2}\mathcal{B}-2\sinh(2\mathcal{J})}\right]. \qquad (2.2.21)$$

• محاسبه مغناطش متوسط

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تعادل انتقالی :

$$\langle \sigma \rangle = \frac{1}{N} \left\langle \sum_{i=1}^{N} \sigma_i \right\rangle$$

$$\langle \sigma \rangle = \frac{1}{N} \sum_{\{\sigma\}} \left(\sum_{i=1}^{N} \sigma_i \right) \frac{1}{Z} e^{\beta J \sum_{i=1}^{N} \sigma_i \sigma_{i+1} + \beta B \sum_{i=1}^{N} \sigma_i}$$

$$\langle \sigma \rangle = \frac{1}{N} \frac{1}{Z} \frac{\partial}{\partial (\beta B)} Z$$

$$\langle \sigma \rangle = \frac{1}{N} \frac{\partial}{\partial (\beta B)} \ln Z$$

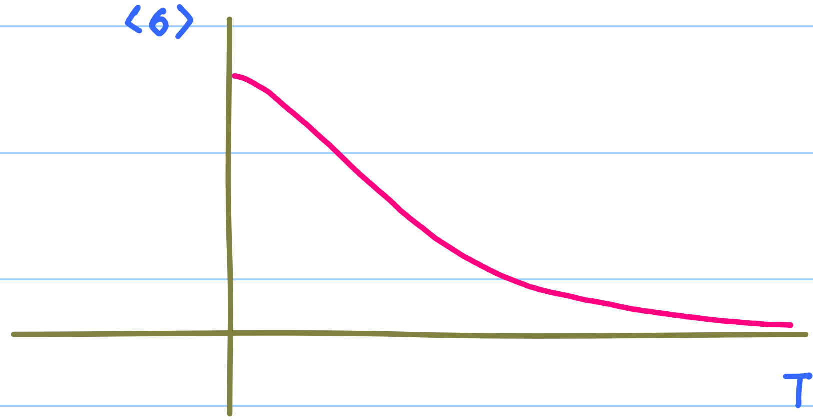
$$\langle \sigma \rangle = \frac{\partial}{\partial (\beta B)} \ln \lambda_+$$

$$\lambda_+ = e^{\beta J} \cosh \beta B + \sqrt{e^{2\beta J} \cosh^2 \beta B - 2 \sinh 2\beta J}$$

$$\langle \sigma \rangle = \frac{e^{\beta J} \sinh \beta B}{\sqrt{e^{2\beta J} \cosh^2 \beta B - 2 \sinh 2\beta J}}$$

1) $\beta \to 0 \quad (T \to \infty) \qquad \langle \sigma \rangle \to 0$

2) $\beta \to \infty \quad (T \to 0) \qquad \langle \sigma \rangle \to 1.$



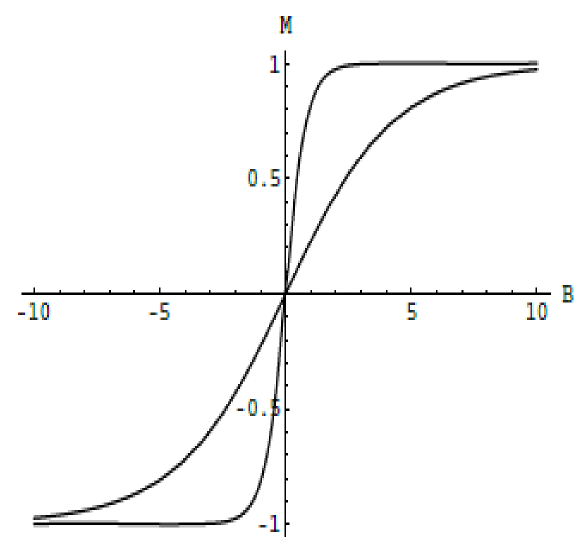
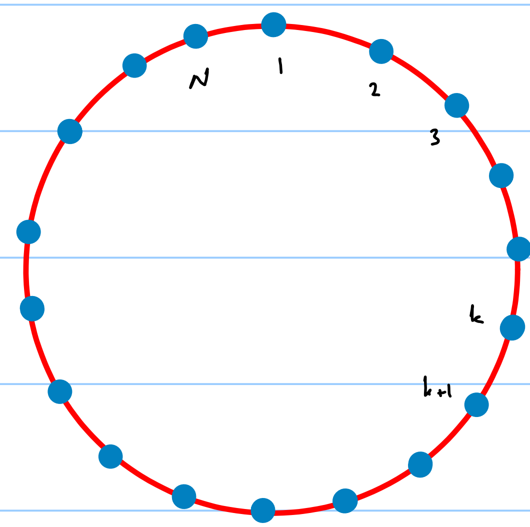


Fig. 2.7 *Magnetization versus the magnetic field B, for different values of the temperature.*

Correlation functions • توابع همبستگی



$$\langle \sigma_m \sigma_n \rangle = \frac{1}{Z}\sum_{\sigma_1 \dots \sigma_N} \sigma_m \sigma_n \, e^{-\beta H}$$

$$= \frac{1}{Z}\sum_{\{\sigma\}} \sigma_m \sigma_n \cdots \langle \sigma_{m-1}|V|\sigma_m\rangle \cdots\cdots \langle \sigma_{n-1}|V|\sigma_n\rangle \cdots$$

$$= \frac{1}{Z}\sum_{\{\sigma\}} \sigma_m \sigma_n \cdots \langle \sigma_{m-1}|V|\sigma_m\rangle \cdots\cdots \langle \sigma_{n-1}|V|\sigma_n\rangle \cdots$$

$$\hat{\sigma}\,|\sigma_m\rangle = \sigma_m\,|\sigma_m\rangle$$

$$\hat{\sigma}\,|1\rangle = |1\rangle$$

$$\hat{\sigma}\,|-1\rangle = -\,|-1\rangle$$

$$\hat{\sigma} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\langle \sigma_m \sigma_n \rangle = \frac{1}{Z} \sum_{\{\sigma\}} \quad \cdots\cdots \langle \sigma_{m-1} |\, V \hat{\sigma}\, | \sigma_m \rangle \cdots \langle \sigma_{n-1} |\, V \hat{\sigma}\, | \sigma_n \rangle \cdots$$

$$\rightarrow \quad \langle \sigma_m \sigma_n \rangle = \frac{1}{Z}\, \mathrm{tr}\left(V^{m}\ \hat{\sigma}\ V^{n-m}\ \hat{\sigma}\ V^{N-n} \right)$$

$$\langle \sigma_m \sigma_n \rangle = \frac{\mathrm{tr}\left(V^m \hat{\sigma} V^{-m} \; V^n \hat{\sigma} V^{-n} \; V^N \right)}{\mathrm{tr}\left(V^N \right)}$$

Define: $\hat{\sigma}_H(m) := V^m \hat{\sigma} V^{-m}$

$$\rightarrow \quad \langle \sigma_m \sigma_n \rangle = \frac{\mathrm{tr}\left(\hat{\sigma}_H(m) \hat{\sigma}_H(n) V^N \right)}{\mathrm{tr}\left(V^N \right)}$$

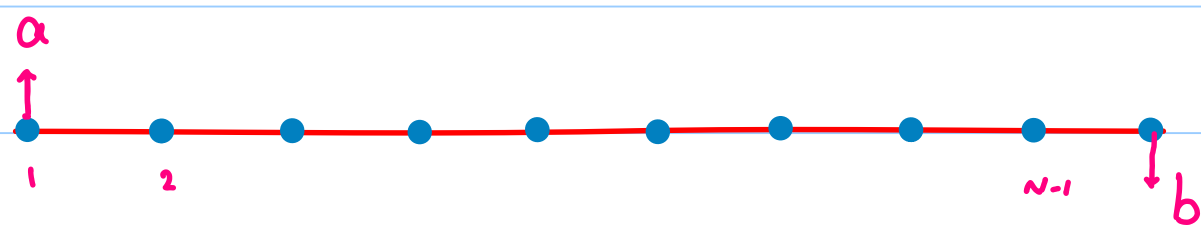
$$V|0\rangle := \lambda_+ |0\rangle \quad \rightarrow$$

$$\langle \sigma_m \sigma_n \rangle = \langle 0| \hat{\sigma}_H(m) \hat{\sigma}_H(n) |0\rangle$$

توابع همبستگی چند نقطه‌ای :

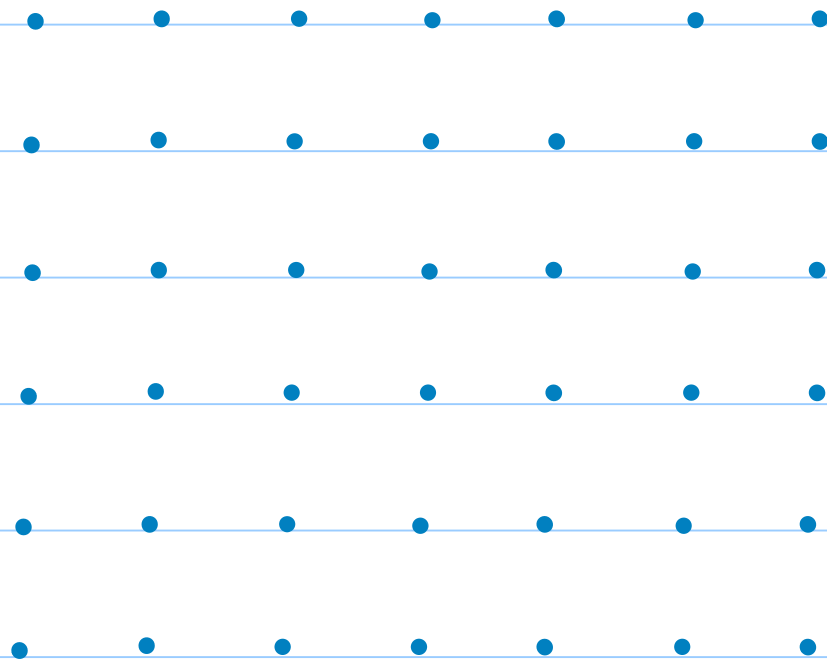
$$\langle \sigma_m \sigma_n \cdots \sigma_p \rangle = \langle 0 | \hat{\sigma}_H(m)\, \hat{\sigma}_H(n) \cdots \hat{\sigma}_H(p) | 0 \rangle$$

Other Boundary conditions :



$$Z_N(a,b) = \langle a | V^{N-1} | b \rangle$$

Method 3) Series Expansion. $B = 0$.



$$Z = \sum_{\{\sigma\}} e^{J \sum_{\text{links}} \sigma_i \cdot \sigma_j}.$$

$$= \sum_{\{\sigma\}} \prod_{\text{links}} e^{J \sigma_i \cdot \sigma_j}.$$

$$e^{J\sigma_i\cdot\sigma_j\cdot} = \cosh J + \sigma_i\cdot\sigma_j\cdot \sinh J$$

$$= \cosh J\,(1 + v\,\sigma_i\cdot\sigma_j\cdot)$$

$$v = \tanh J = \tanh \frac{J}{kT}$$

if $kT \gg J$ $\quad v \ll 1 \quad$ دمای بالا

$$Z = \sum_{\{\sigma\}} e^{J\sum_{\text{links}} \sigma_i\cdot\sigma_j\cdot}$$

$$= \sum_{\{\sigma\}} \prod_{\text{links}} (\cosh J + \sigma_i\cdot\sigma_j\cdot \sinh J)$$

$$= \sum_{\{\sigma\}} \prod_{\text{links}} (\cosh J)\,(1 + v\,\sigma_i\cdot\sigma_j\cdot)$$

$$Z = (\cosh J)^{|\text{links}|} \underbrace{\sum_{\{\sigma\}} \prod_{\text{links}} (1 + v\,\sigma_i \sigma_j)}_{Q.}$$

$$Q = \sum_{\{\sigma\}} (1 + v\,\sigma_1 \sigma_2)(1 + v\,\sigma_2 \sigma_3) \cdots\cdots$$

$$Q = Q_0 + v\,Q_1 + v^2 Q_2 + \cdots$$

$d = 1 \quad \rightarrow \quad |\text{links}| = N$

$d = 2 \quad \rightarrow \quad |\text{links}| = 2N$

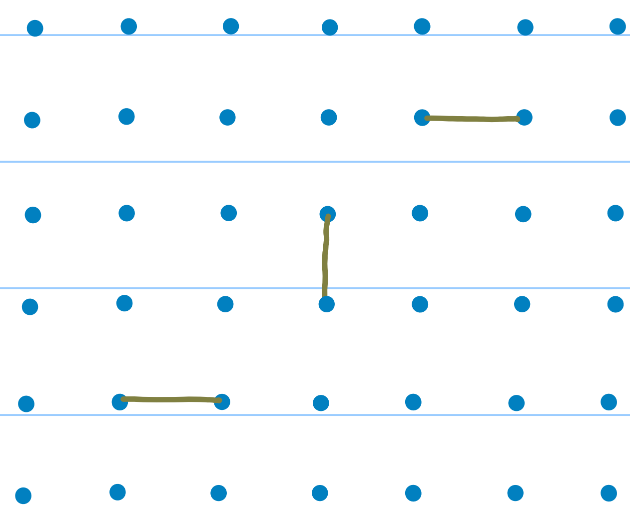
$T \rightarrow \infty$ ترتیب اول.

$$Q_0 = \sum_{\{\sigma\}} 1 = 2^N$$

$$Q_1 = ?$$

$$Q = \sum_{\{\sigma\}} (1 + v\sigma_1\sigma_2)(1 + v\sigma_2\sigma_3)\cdots$$

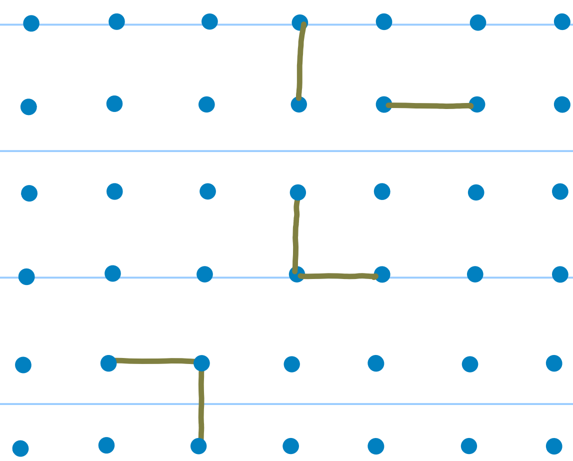
$$= \sum_{\{\sigma\}} 1 + v(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \cdots)$$

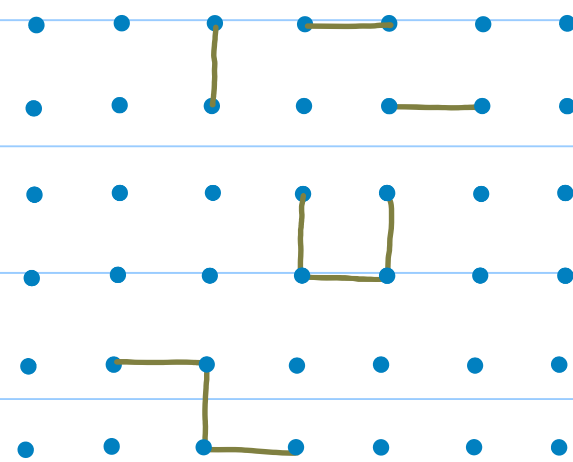


But: $\sum_{\sigma_1} \sigma_1 = 1 + (-1) = 0$

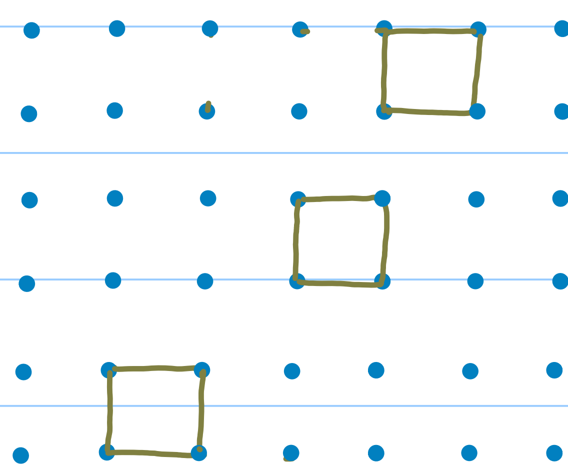
$Q_1 = 0$

For the same reason $Q_2 = 0$





$Q_2 = 0$ $\qquad$ $Q_3 = 0$

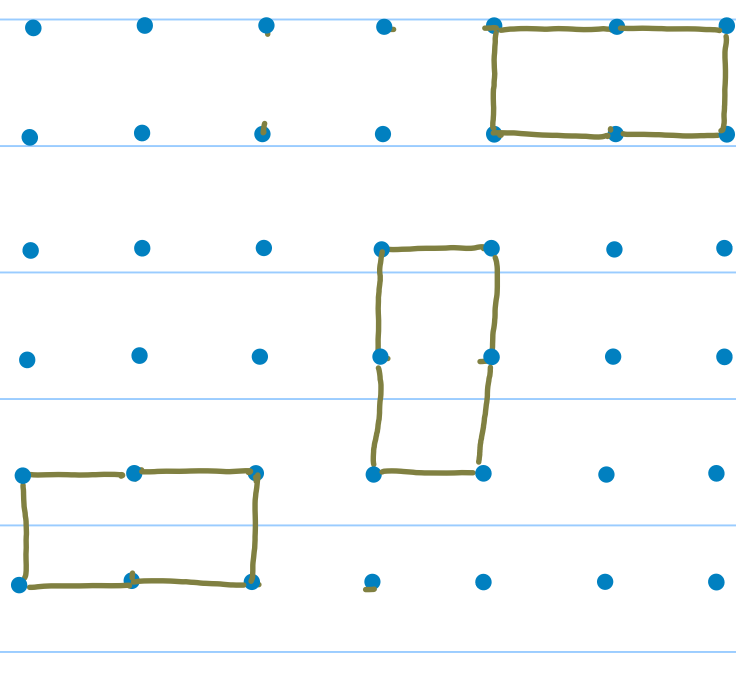


$$Q = \sum_{\{\sigma\}} (1 + v\,\sigma_1\sigma_2)(1 + v\,\sigma_2\sigma_3)\cdots$$

$$Q_4 = v^4 \sum_{\{\sigma\}} \square$$

$$= v^4\, 2^N$$ (تعداد مربع های داخل شبکه)

$$= v^4\, 2^N\, N \qquad\qquad Q_5 = 0$$



$$Q_6 = v^6\, 2^N\,(2N)$$

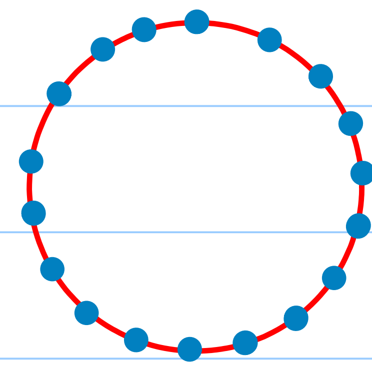
$$\boxed{Z_N = (\cosh J)^{2N}\, 2^N \left\{ 1 + v^4 N + v^6\, 2N + \dots \right\}}$$

$$Z = (\cosh J)^{|\text{links}|}\, 2^N \sum_{l=0}^{\infty} v^l\, n_l$$

تعداد منحنی‌های بسته با طول l .

d=1 →

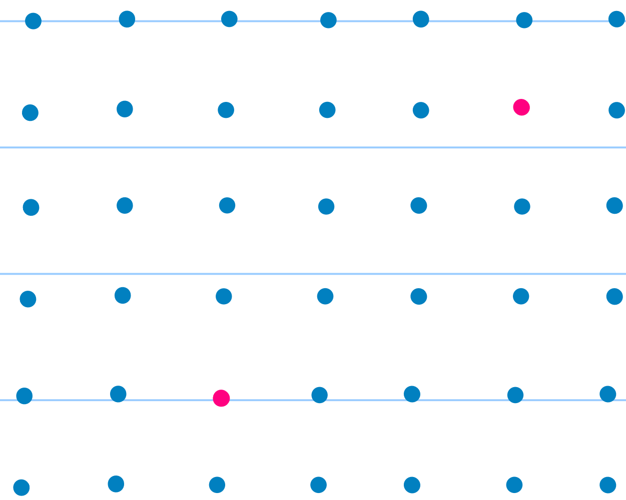
$$Z_N = (\cosh J)^N \, 2^N \left\{ 1 + v^N \right\}$$



$$Z_N = 2^N \left\{ (\cosh J)^N + (\sinh J)^N \right\}$$

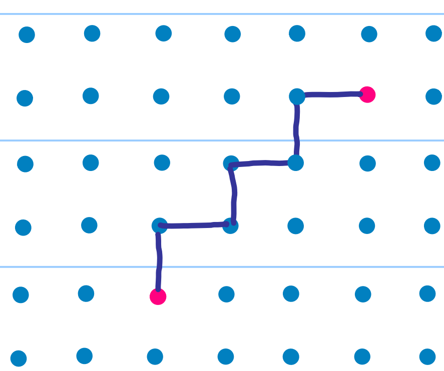
$N \to \infty$ $\qquad Z_N \to (2\cosh J)^N$

Correlation Functions



$$\langle \sigma_m \sigma_n \rangle = \frac{1}{Z} (\cosh J)^{|\text{links}|} \sum_{\{\sigma\}} \sigma_m \sigma_n$$

$$(1 + v\sigma_1\sigma_2)(1 + v\sigma_2\sigma_3) \ldots (1 + v\sigma_N\sigma_1)$$



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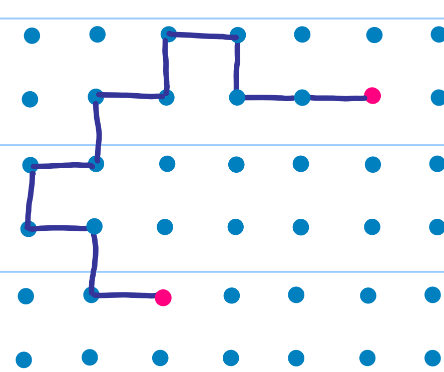
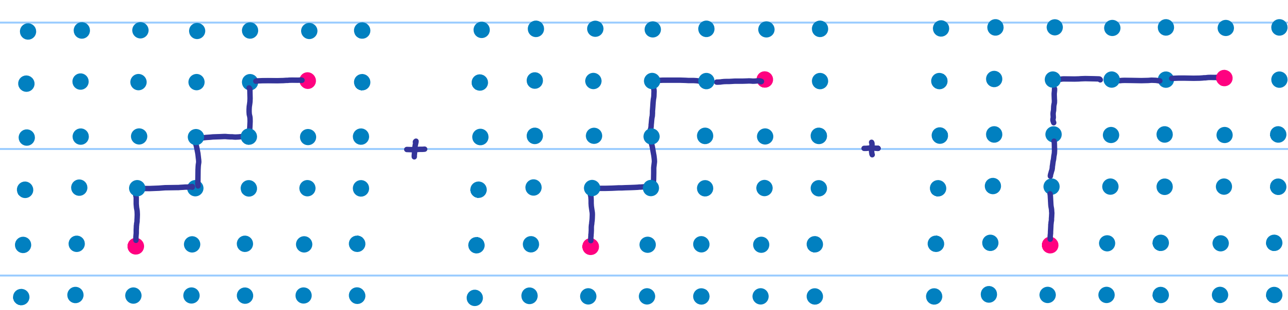


$$\langle \sigma_x \sigma_y \rangle = \frac{\sum_{l=0}^{\infty} v^l \, n_l(x,y)}{\sum_{l=0}^{\infty} v^l \, n_l}$$

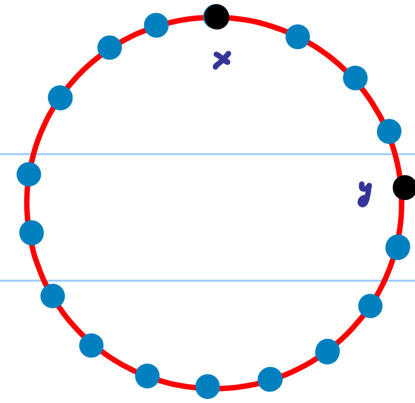
$n_l(x,y) =$ تعداد مسیرهای با طول l که x را به y وصل می‌کنند

$$\langle \sigma_x \sigma_y \rangle = \frac{\alpha \, v^{|x-y|} + \cdots}{1 + v^4 N + v^6 2N + \cdots} \sim \alpha v^{|x-y|}$$



$\downarrow$

α



$$\langle \sigma_x \sigma_y \rangle = \frac{\left\{ v^{x-y} + v^{N-(x-y)} \right\}}{\left\{ 1 + v^N \right\}}$$

$$\langle \sigma_x \sigma_y \rangle = \frac{v^{x-y} + v^{N-(x-y)}}{1 + v^N}$$

$N \to \infty$ $$\langle \sigma_x \sigma_y \rangle = v^{|x-y|}$$

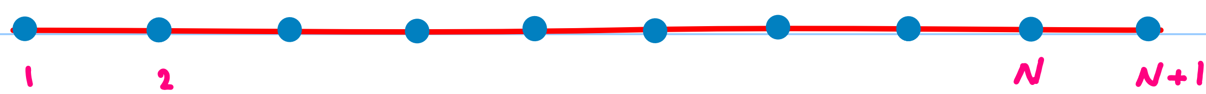
$$= (\tanh \beta J)^{|x-y|}$$

The Potts Model

$$\sigma_i = 1, 2, 3, \ldots\ldots q$$

$$\mathcal{H} = -J \sum_{i,j} \delta(\sigma_i, \sigma_j)$$

1) Recursive Method



$$Z_{N+1} = \sum_{\{\sigma\}_{N+1}} e^{J \sum_{i=1}^{N-1} \delta(\sigma_i, \sigma_{i+1}) + J \delta(\sigma_N, \sigma_{N+1})}$$

$$Z_{N+1} = \sum_{\{\sigma\}_N} e^{J \sum_{i=1}^{N-1} \delta(\sigma_i, \sigma_{i+1})} \sum_{\sigma_{N+1}} e^{J\delta(\sigma_N, \sigma_{N+1})}$$

$$\sum_{\sigma_{N+1}} e^{J\delta(\sigma_N, \sigma_{N+1})} = e^J + (q-1)$$

$$Z_{N+1} = Z_N \left(e^J + (q-1) \right)$$

$$\rightarrow \quad Z_N = \left(e^J + (q-1)\right)^{N-1} Z_1$$

or

$$\boxed{Z_N = q \prod_{i=1}^{N-1} \left(e^{J_i} + (q-1) \right).}$$

$$Z_N = q\left(e^J + (q-1)\right)^{N-1}$$

Transfer Matrix — بخش دوم: ماتریس انتقال.

$$Z_N = \sum_{\{\sigma\}_N} e^{J \sum_{i=1}^{N} \delta(\sigma_i, \sigma_{i+1})}$$

$$= \operatorname{tr}(V^N) \qquad \langle \sigma | V | \sigma' \rangle = e^{J \delta(\sigma, \sigma')}$$

$$V = \begin{vmatrix} x & 1 & 1 & 1 & 1 \\ 1 & x & 1 & 1 & 1 \\ 1 & 1 & x & 1 & 1 \\ 1 & 1 & 1 & x & 1 \\ 1 & 1 & 1 & 1 & x \end{vmatrix} \qquad x = e^{J}$$

$$V = (x-1) I + |\psi\rangle\langle\psi|$$

$$|\psi\rangle = \begin{bmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \qquad \langle\psi| = [1 \ 1 \ 1 \ \dots \ 1]$$

$$\langle\psi|\psi\rangle = q$$

$$V|\psi\rangle = x|\psi\rangle + q|\psi\rangle = (e^J + q)|\psi\rangle$$

$$V|\psi^\perp\rangle = x|\psi^\perp\rangle$$

$$\rightarrow \quad Z_N = tr(V^N) = \lambda_1^N + \lambda_2^N + \dots \lambda_q^N$$

$$= (x+q)^N + (q-1)x^N$$

Thermodynamic limit:

$$Z_N \longrightarrow Z = (e^J + q - 1)^N$$

$$f = \frac{F}{N} = -kT \ln\left(e^{J} + q - 1\right)$$

Series Expansion   روش سوم: بسط سری

$$Z = \sum_{\{\sigma\}} \prod_{\text{links}} e^{J\,\delta(\sigma,\sigma')}$$

$$e^{J\,\delta(\sigma,\sigma')} = \begin{cases} e^{J} & \text{if} \quad \sigma = \sigma' \\ 1 & \text{if} \quad \sigma \neq \sigma' \end{cases}$$

$$e^{J\,\delta(\sigma,\sigma')} = e^{J}\,\delta_{\sigma,\sigma'} + \left(1 - \delta_{\sigma\sigma'}\right)$$

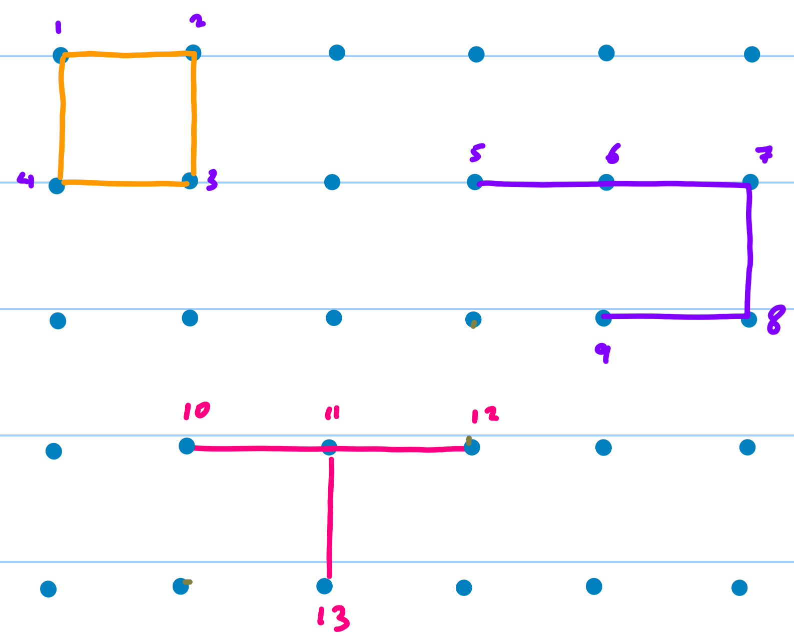
$$= 1 + \left(e^{J} - 1\right)\delta_{\sigma,\sigma'}$$

$$= 1 + v\,\delta_{\sigma\sigma'}$$

$$T \to \infty \quad \to \quad \beta \to 0 \quad \to \quad v \to 0$$

$$Z = \sum_{\{\sigma\}} \prod_{\text{links}} \left(1 + v\,\delta_{\sigma\sigma'}\right)$$

$$Z = \sum_{\{\sigma\}} (1+v\,\delta_{\sigma_1\sigma_2})(1+v\,\delta_{\sigma_2\sigma_3})\dots(1+v\,\delta_{\sigma_k\sigma_\ell})\dots$$



$$(v^4\,\delta_{12}\,\delta_{23}\,\delta_{34}\,\delta_{41})(v^4\,\delta_{56}\,\delta_{67}\dots\delta_{89})(v^2\,\delta_{10,11}\,\delta_{11,12}\,\delta_{11,13})$$

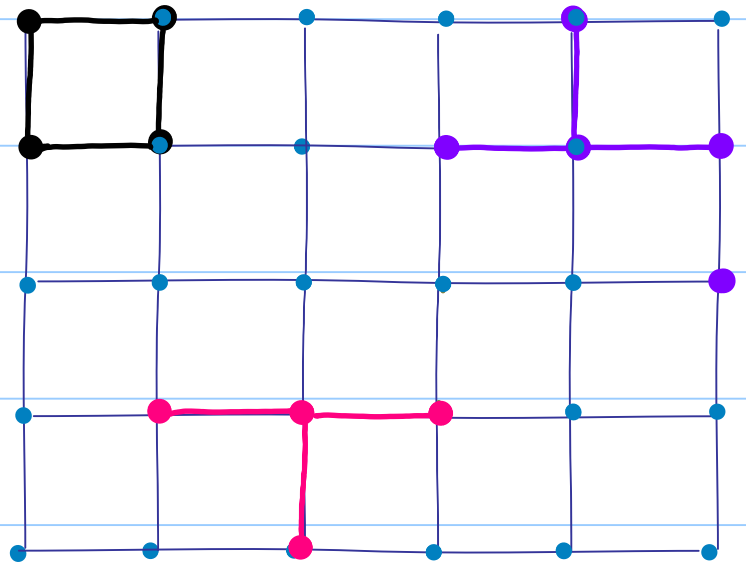
$$\sum_{\{\sigma\}} \left(v^{4}\,\delta_{12}\,\delta_{23}\ldots\delta_{41}\right)\left(v^{4}\,\delta_{56}\ldots\ldots\delta_{89}\right)\left(v^{2}\,\delta_{10,11}\,\delta_{11,12}\right)$$

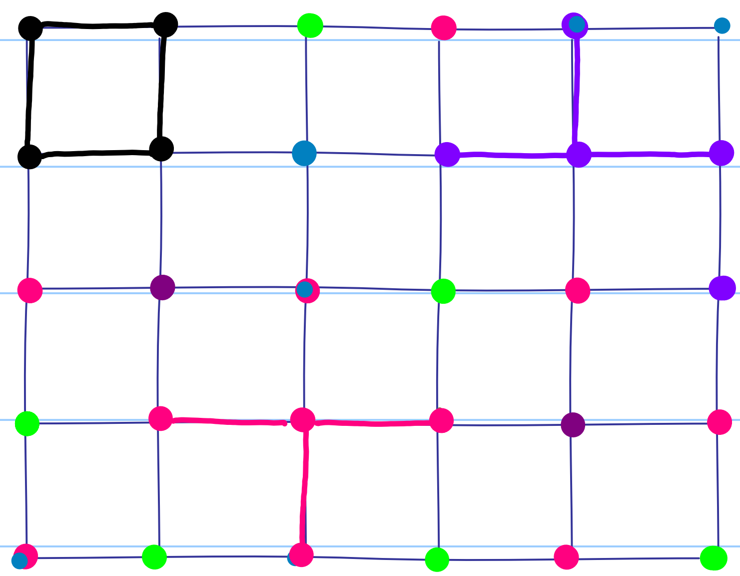
$$\rightarrow \quad v^{4+4+2}\, q^{C} = v^{l}\, q^{C}$$

C = تعداد گراف ۶ .

$$Z = \sum_{\text{graphs}} v^{l}\, q^{C}$$

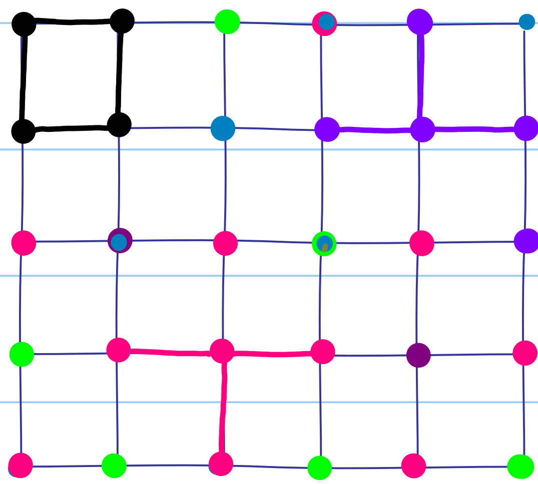
$$Z_N = q^{N} + v\, q^{N-1}(2N) + \ldots$$

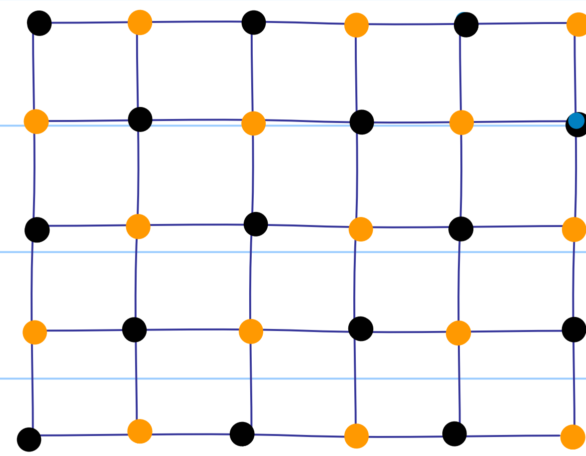




if $J \to -\infty$ $\quad e^{J\delta_{\sigma_i \sigma_j}} \longrightarrow \begin{cases} 0 & \text{if } \sigma_i = \sigma_j \\ 1 & \text{if } \sigma_i \neq \sigma_j \end{cases}$

بنابراین در حد $J \to -\infty$ ، تنها حالت‌هایی در تابع پارش باقی خواهند ماند که تمام نقاط مجاور رنگ‌های مختلف داشته باشند.





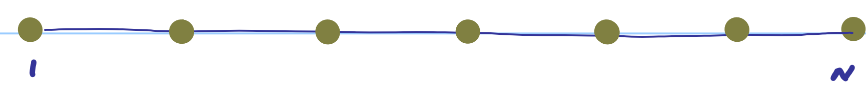
0

با $J \to -\infty$

$$Z(J \to -\infty, q) = Z(v \to -1, q) = P_G(q)$$

$P_G(q) =$ تعداد راه‌هایی که می‌توان با q رنگ گراف G را رنگ زد به نحوی که نقاط همسایه هم‌رنگ نباشند.

شکل:

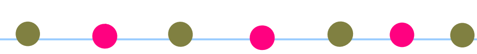


$$Z = q\,(e^{J}-1+q)^{N-1}$$

$$P(q) = q\,(q-1)^{N-1}$$

$P(1) = 0$

$P(2) = 2$



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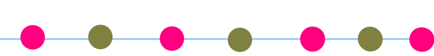
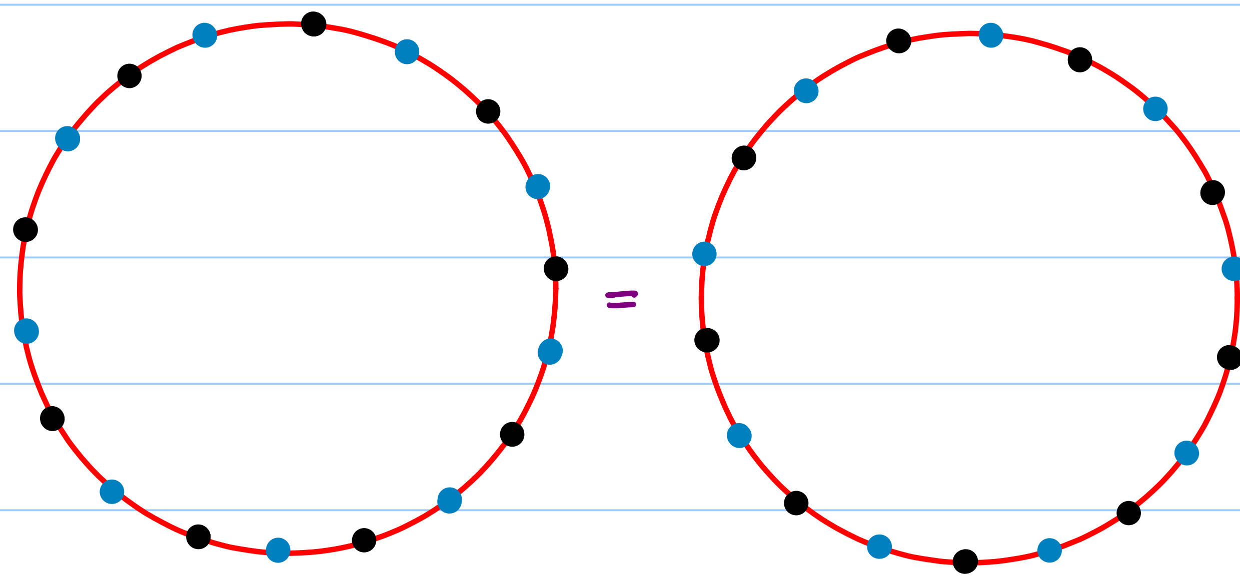


$P(3) = 3 \times 2^{N-1}$

Interesting Point : For the Ring:

$$Z = (e^{J}-1+q)^{N}$$

$\rightarrow \quad P(q) = (q-1)^N \quad \rightarrow \quad P(2) = 1$



مسئله k - رنگ پذیری

if $P_G(k) = 0 \rightarrow$ نمی‌توان گراف G را با k رنگ، رنگ آمیزی کرد.

Conjecture: For any planar Graph

$$P_G(4) \neq 0.$$